Topic-16

Von Neumann Entropy

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1 Introduction

In the last lecture, we discussed Shanon entropy and the limit it imposes on uniquely decipherable codes. It was seen that the maximum length of a word cannot exceed the Shannon entropy calculated using the probability of occurrence of a letter in the code. Von Neumann provided a generalization of the above to the case of quantum systems by defining a quantum entropy which bears his name. Like the case that Shannon entropy is a measure of uncertainty associated with a classical event, Von Neumann entropy provides a measure of ignorance of a quantum system.

2 Von Neumann Entropy

Consider a quantum system described by a density matrix ρ. The expectation value of a physical quantity described by a quantum mechanical operator A is given by the weighted average of the expectation value of the operator in different quantum states that constitute the ensemble,

$$\langle A\rangle = \text{Tr } (A\rho)$$

subject to the condition $\text{Tr } (\rho) = 1$. For a pure system $\text{Tr } (\rho^2) = 1$ as well. Von Neumann entropy provides a measure of the degree of mixedness of the system. For a pure system, the Von Neumann entropy $S(\rho) = 0$ should be zero. This is similar to the classical case in which the probability of the event being 1 or 0 for which the Shannon entropy is zero. Taking the cue from Shannon, Von Neumann defined the entropy of a quantum system to be given by the expression

$$S(\rho) = -\text{Tr } (\rho \log_2 \rho)$$

Let us summarize the properties of the entropy defined thus.

1. Since the trace is basis independent, the definition of entropy implies that it is independent of the basis in which the quantum states are expressed. Thus if we go over to a basis in which the density matrix is diagonal, we have

$$S(\rho) = -\sum_i \lambda_i \log_2(\lambda_i)$$

Since the density matrix is a positive matrix with trace being equal to 1, the eigenvalues are $0 \leq \lambda_i \leq 1$ and $\sum_i \lambda_i = 1$. This implies that entropy is a positive quantity.

2. For a pure state, only one eigenvalue of ρ is 1 and all others are zero. This implies $S(\rho) = 0$ for a pure state.

3. We had shown earlier (see Notes accompanying Lecture 35) that as a consequence of Gibb's inequality, one can show that if there are D non-zero eigenvalues , the maximum value of entropy function occurs when all the eigenvalues are equal, i.e. each eigenvalue is $1/D$ so that maximum entropy is $\log_2 D$, i.e. $S(\rho) \leq \log_2 D$ with the equality when all eigenvalues are equal.

4. Concavity : For $x_i \geq 0$ and $\sum_i x_i = 1$, we have $S(\sum_i x_i\rho_i) \geq \sum_i x_i S(\rho_i)$, which is the quantum equivalent of mixing theorem for entropy, i.e. the less we know about how the state was prepared higher will be the entropy.

5. Sub-Additivity: For a bipartite system AB with ρ^{AB} as its density matrix, we have

$$S(A, B) \leq S(A) + S(B)$$

with the equality being valid for the case for which $\rho^{AB} = \rho^A \otimes \rho^B$, i.e. if the systems are independent.

6. Conditional entropy $S(A|B)$ is defined as

$$S(A|B) = S(AB) - S(B)$$

which signifies the amount of uncertainty B(Bob) still has about the state of A(Alice).

Proof of these theorems depend on Klein's inequality, which do not have time to go for here).

We will illustrate some of the properties stated above. For pure states, the entropy is zero because there is only one non-zero eigenvalue which is 1. Thus ρ^A and ρ^B both have zero entropy as does ρ^{AB}.

Consider a state $\frac{1}{\sqrt{2}}(\mid 0\rangle + \mid 1\rangle)$. The density matrix corresponding to it has the representation $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. The eigenvalues can be easily be seen to be 0 and 1 , so that the entropy is zero.
Consider the maximally mixed state $\rho = \frac{1}{2}(\mid 0\rangle\langle 0 \mid + \mid 1\rangle\langle 1 \mid)$. The matrix representation is $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, which have eigenvalues 1/2 and 1/2 so that the entropy is $-(1/2)\log_2(1/2) - 1/2\log_2(1/2) = 1$.
Consider an example to illustrate sub-additivity. Consider a state $\cos\theta \mid 00\rangle + \sin\theta \mid 11\rangle$. The state is pure with entropy equal to zero. Consider now the partial entropy of the first particle (A). The reduced density matrix for this is obtained by taking the partial trace over B of the density matrix

$$\rho^A = \mathrm{Tr_B}\, \rho^{AB} = \cos^2\theta \mid 0\rangle\langle 0 \mid + \sin^2\theta \mid 1\rangle\langle 1 \mid$$

The eigenvalues are $\cos^2\theta$ and $\sin^2\theta$ so that the entropy is

$$S(A) = -2\cos^2\theta\log(\cos\theta) - 2\sin^2\theta\log(\sin\theta)$$

An identical expression is valid for $S(B)$. If $\theta \neq 0$, $\sin^2\theta$ and $\cos^2\theta$ are each less than 1 and hence $S_A > 0$ and $S_B > 0$. Thus sub-additivity is valid.