Topic -17
EPR and Bell's Inequalities

Dipan Kumar Ghosh
Indian Institute of Technology Bombay Powai, Mumbai 400076

April 15, 2017

1 Introduction

Till now we have been discussing various aspects of quantum computing. We would end this series by a few few lectures on quantum cryptography, an area which holds maximum promise for possible applications. We had earlier pointed out that even such distinguished scientists as Einstein had reservations about the Copenhagen interpretation of quantum mechanics. However, before we do that, one would like to ask how robust are the quantum mechanical principle on which these are based. In 1935, Einstein, Podolsky and Rosen published a paper which essentially asks the question : "can quantum mechanical description of physical reality be considered to be complete?" This seminal paper goes by the name EPR paradox in physics.

To get an idea about the objections posed by EPR, let us look at the phenomenon of entanglement. In these lectures, we have given several examples of entanglement. Take the most well known set of 4 states known as the Bell states. For definitiveness consider the state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$, which is maximally entangled, which implies that if we compute the reduced density matrix of either of the constituent qubits, we would get a mixed state $\rho_{A/B} = \frac{I}{2}$. What it means is that when we make a measurement of either the first or the second qubit, we would get a maximally mixed state which does not give any idea about how the original state was prepared because a measurement of the first qubit also collapsed the second qubit as well (and vice versa). Herein lies Einstein's objection. If two objects are separated by a far enough distance such that no information about a measurement of the first qubit can reach the location of the second qubit during the time scale required for the measurement, it is inconceivable that the measurement of the first qubit determines the state of the second qubit. This has been known as "spooky action at a distance" correlation between state of two particles irrespective of how widely they are separated. Einstein provided a gedanken experiment which points towards what Einstein thought is an incomplete description of the system as per quantum mechanics. According to Einstein, a system must have an "element of physical reality". In Einstein's own words "If, without in any way disturbing the system, we can predict with certainty (i.e. with a probability of one), the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity".

How does it differ from what we say in quantum mechanics? In quantum mechanics, the value of a physical quantity has no meaning independent of its measurement (i.e. observation). Let us consider a particle of total spin angular momentum $S = 0$ decaying into two particles of spin half each. The total angular momentum of the two particle system should still be $S = 0$, i.e. the pair of particles will be in the state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$, i.e. if the S_z value of particle 1 is $\uparrow$ (i.e. $S_z = +\hbar/2$), that of the second particle is $\downarrow$, i.e. it has $S_z = -\hbar/2$ and vice versa. Let us now suppose after the decay the two particles separate and get separated by a large distance so that any disturbance of one particle cannot influence the state of the other particle. Suppose, Alice has the first particle and Bob the second. Suppose Alice measures the spin state of her particle and finds it to be in $\uparrow$ state, i,e. in the state $S_z = +\hbar/2$. What quantum mechanics says is that, if now, Bob measures the spin state of the particle that he has got, he is guaranteed to measure it as $-\hbar/2$. So we notice that Alice's measurement is perfectly anti-correlated with that of Bob.

If however, Alice and Bob measure their spin components in different directions, the result will not lead to a perfect anti-correlation. Suppose Alice, as before, measure the z-component of her spin and got a result $S_z = +1/2$, i.e. $\sigma_z = +1$. Bob, instead of measuring the spin of his particle in the z-direction, measures it along an arbitrary direction having a unit vector $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. The result of his measurement would depend on the angle that his axis of measurement makes with the z-axis. We had seen earlier that while the states which are eigenstates of σ_z are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ corresponding to the eigenvalues ± 1, the eigenstates of σ_n are

$$| \hat{n}, +\rangle = e^{i\varphi/2}\begin{pmatrix}\cos(\theta/2)e^{-i\varphi/2}\\ \sin(\theta/2)e^{i\varphi/2}\end{pmatrix}$$

corresponding to $\sigma_n = +1$ and

$$| \hat{n}, -\rangle = e^{i\varphi/2}\begin{pmatrix}\sin(\theta/2)e^{-i\varphi/2}\\ -\cos(\theta/2)e^{i\varphi/2}\end{pmatrix}$$

corresponding to $\sigma_n = -1$.

Let us look at the rival points of view. According to quantum mechanics, the particle does not have properties independent of observation, one can only assign probabilities of possible values that will be obtained on measuring a physical property. The measurement process itself gives the result. The rival theory known as the "Hidden variable theory" claims that though the process of measurement reveals the value of a physical property, the result was already pre-ordained as it was coded into the system through some 'hidden

variable' that is hidden from our eyes and from our understanding.
To resolve this dichotomy John Bell proposed another gedanken experiment (i.e. a thought experiment). Consider any of the four Bell states, e.g. the state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. For clarity, we denote particle 1 by A (for Alice's particle) and particle 2 by B (for Bob). Consider the i-th component of the total spin i.e. $\sigma_i^A + \sigma_i^B$. It is easily show that the expectation value of this operator in the Bell state above is zero. For instance, consider $i = x$, i.e. the operator $\sigma_x^A + \sigma_x^B$. We have

$$(\sigma_x^A + \sigma_x^B)(|10\rangle - |01\rangle) = [(|00\rangle - |11\rangle) + (|11\rangle - |00\rangle)] = 0$$

as σ_x operator flips the spin. This can be verified for any of the components.
This result shows that if Alice measures her spin along some direction a and Bob measures it along some direction b. Because of the result that $\sigma_i^A + \sigma_i^B = 0$, if Bob reveals the result of his measurement, Alice would know what result she would have got, had she made a measurement along Bob's direction, i.e. just the negative of what Bob got.

According to the standard (Copenhagen) interpretation of quantum mechanics, a particle does not have physical properties independent of observation; they arise as a result of observation, quantum mechanics simply gives the probabilities of different possible results for a given state of the system under observation. According to EPR, the explanation of Bob's result being dependent on Alice's measurement which preceded it, arises out of the fact that the result was pre-ordained because the property of the system, though revealed during the process of measurement, was actually encoded within the system in some hidden variables. This was because of EPR's element of *physical reality and locality*, which the quantum mechanics fails to incorporate thereby making it an incomplete theory. The argument also assumes that the interaction between the measuring apparatus and the system being measured is *local* whose effect is confined to the place where the measurement is made and it does not have any influence on objects at a distant location. In 1964 John Bell proposed derived an inequality (of which there are several versions by various authors) which must be satisfied by measurement processes, if there existed hidden variables in the system. Violation of these inequalities would then rule out the existence of hidden variables and would provide credence to an alternate theory (in this case quantum mechanics). Several attempts have been made, notably by David Bohm, to construct such hidden variable theories but such theories turn out to be contrived. For instance, in order to explain single electron diffraction in a typical double slit experiment, it would be necessary to assume that though an electron can pass only through one of the slits, one will have to assume a special force that acts on the electron only when the other slit is open. The Copenhagen interpretation is of the view that indeterminacy observed in nature is fundamental and does not in any way reflect our inadequacy of scientific knowledge.

A way out of the apparent conflict is to assume that the second measurement by which the first particle gets a definite position, somehow prevents the second particle from being in a state of definite momentum in spite of the fact that the two particles might be separated by large distance. The two particles continue to remain part of a single system, i.e. the states of the two particles are **entangled**.

2 Bell states and Local measurement

John Bell proposed a test for verifying whether EPR's idea of a hidden variable is indeed correct. He did it by considering a set of entangled states known as the **Bell states** which are given by

$$\varphi^+_{AB} = \frac{1}{\sqrt{2}}(\mid 0_A 0_B\rangle + \mid 1_A 1_B\rangle) \tag{1a}$$

$$\varphi^-_{AB} = \frac{1}{\sqrt{2}}(\mid 0_A 0_B\rangle - \mid 1_A 1_B\rangle) \tag{1b}$$

$$\psi^+_{AB} = \frac{1}{\sqrt{2}}(\mid 0_A 1_B\rangle + \mid 1_A 0_B\rangle) \tag{1c}$$

$$\psi^-_{AB} = \frac{1}{\sqrt{2}}(\mid 0_A 1_B\rangle - \mid 1_A 0_B\rangle) \tag{1d}$$

The label A (B) indicates that the particle (spin) belongs to Alice (Bob). It is conventional to classify these states into a parity bit, i.e. are the two bits in the state parallel or antiparallel (this distinguishes between φ states in which the two qubits in each constituents are parallel with ψ in which they are antiparallel. On the other hand, we can classify them as per the phase bit, which tells us whether the relative sign is positive or negative.

These states are **maximally entangled**. Recall that an entangled state has a Schmidt number greater than 1. Let $\mid \psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a pure state.The state is said to be entangled if one *cannot* find $\mid \phi_A\rangle \in \mathcal{H}_A$ and $\mid \chi_B\rangle \in \mathcal{H}_B$ such that one can express $\mid \psi_{AB}\rangle = \mid \phi_A\rangle \otimes \mid chi_B\rangle$. In such a case the Schmidt sium has at least two terms. If $\dim \mathcal{H}_A \leq \dim \mathcal{H}_B$, then a state is maximally entangled if $\rho_A = \frac{1}{\dim \mathcal{H}_A}$. The Bell states mentioned above have Schmidt rank of 2. It is easy to check that the reduced density operator is $I/2$ which means that if we measure the state of the qubit A along any axis, we would get spin up or spin down with equal probability which does not give any information on how the state was prepared. However, once the state of the qubit A is measured the result of the measurement of B is definite. The measurement gives no information on how the state was prepared. This contrasts with the situation for one qubit, where in view of (**??**) one can store a bit by preparing along an axis $\hat{n}$ and recover how it was prepared by measuring along that axis. In the maximally entangled case, however, measurement of two qubits cannot reveal information about how it was prepared. (The von-Neumann entropy of a maximally entangled state is maximum.) Bob and Alice can do local manipulation of his or her qubit but by doing so they can only change from one maximally entangled state to another; they cannot change that reduced density matrix of the system. For instance Alice can apply σ_z to her bit which results in converting a $+$ to a $-$ while retaining φ or ψ as the case may be. On the other hand, she may apply σ_x to her bit which would interchange φ with ψ (it also changes the phase bit if applied to $-$ states).

Suppose both Alice and Bob prepare their states as simultaneous eigenstates of $\sigma_z^A\sigma_z^B$ and $\sigma_x^A\sigma_x^B$. It is easy to check that these two operators commute:

$$\begin{aligned}
[\sigma_z^A\sigma_z^B, \sigma_x^A\sigma_x^B] &= \sigma_z^A\sigma_x^A[\sigma_z^B, \sigma_x^B] - [\sigma_z^A, \sigma_x^A]\sigma_z^B\sigma_x^B \\
&= 2i\sigma_z^A\sigma_x^A\sigma_y^B - 2i\sigma_y^A\sigma_z^B\sigma_x^B \\
&= -2\sigma_y^A\sigma_y^B + 2\sigma_y^B\sigma_y^A = 0
\end{aligned}$$

As these operators commute, one can do a simultaneous measurement. However they cannot do a local measurement. For instance, both Alice and Bob could measure their spins along z-axis, thereby getting information about σ_z^A and σ_z^B. Since each of these operators commute with $\sigma_z^A\sigma_z^B$, they can get information about $\sigma_z^A\sigma_z^B$. For instance, if Alice got $\mid 0\rangle$ and Bob got $\mid 1\rangle$, the state must be one of the two states $\psi^\pm$. Alternatively, if both got $\mid 0\rangle$ or $\mid 1\rangle$, the state must have been one of the $\varphi^\pm$. However, there is no way they can get information about the phase bit, i.e., is it a $+$ or a $-$? This is because σ_z^A or σ_z^B does not commute with $\sigma_x^A\sigma_A^B$ and hence, the phase bits being eigenstates of $\sigma_x^A\sigma_z^B$, cannot be determined from such local measurement. Similarly, local measurement by σ_x^A and σ_x^B can give information about the phase bit but not about the parity bit. However, if Alice and Bob work jointly, they can convert the Bell basis to a computational basis and get appropriate information regarding the state they started from.

3 Bell's Inequalities

Let us consider the correlation between the measurement made by Alice and Bob as per predictions of quantum mechanics. Consider the state $\psi^- = \frac{1}{\sqrt{2}}(\mid 01\rangle - \mid 10\rangle)$. It can be seen that this state gives zero when acted upon by $\sigma_i^A + \sigma_i^B$. For instance, if $i = z$ or x,

$$\begin{aligned}
(\sigma_z^A + \sigma_z^B)\frac{(\mid 01\rangle - \mid 10\rangle)}{\sqrt{2}} &= \frac{(\mid 01\rangle + \mid 10\rangle)}{\sqrt{2}} + \frac{(-\mid 01\rangle - \mid 10\rangle)}{\sqrt{2}} = 0 \\
(\sigma_x^A + \sigma_x^B)\frac{(\mid 01\rangle - \mid 10\rangle)}{\sqrt{2}} &= \frac{(\mid 11\rangle + \mid 00\rangle)}{\sqrt{2}} + \frac{(-\mid 00\rangle - \mid 11\rangle)}{\sqrt{2}} = 0
\end{aligned}$$

Consider Alice measuring along an axis n and Bob measuring about an axis m. It can be seen that

$$\begin{aligned}\langle\psi^- \mid (\sigma^A\cdot\hat{n})(\sigma^B\cdot\hat{m}) \mid \psi^-\rangle &= -\langle\psi^- \mid (\sigma^A\cdot\hat{n})(\sigma^A\cdot\hat{m}) \mid \psi^-\rangle \\ &= -\langle\psi^- \mid \sum_{i,j} n_i\sigma_i^A m_j \hat{n}\sigma_j^A \mid \psi^-\rangle \\ &= -\frac{1}{2}\sum_{i,j} n_i m_j \text{tr}(\sigma_i^A\sigma_j^A) \\ &= -\sum_{i,j} n_i m_j \delta_{i,j} = -\hat{n}\cdot\hat{m} = -\cos\theta \qquad (2)\end{aligned}$$

In deriving the above, we have in the first line replaced σ^B by $-\sigma^A$ as the state concerned yields zero on being acted on by $(\sigma^A+\sigma^B)$. In the third line, we have substituted the expression for ψ^- and as the operators have no reference to B, we have used the normalization property. It may be observed that the measurement is always anti-correlated, even when the axes chosen by Alice and Bob are different. The anti-correlation is perfect if the axes are the same so that $\cos\theta = -1$.
Let us look at the expectation value of different results by Alice and Bob when they choose different axes for measurement. It may be noted that the operator $\frac{1}{2}(1\pm\hat{n}\cdot\sigma)$ projects spin up (or down) along the direction of eigenstates of σ_n, given by

$$\mid n,+\rangle = \begin{pmatrix}\cos\frac{\theta}{2}e^{-1\varphi/2} \\ \sin\frac{\theta}{2}e^{i\varphi/2}\end{pmatrix} \qquad (3)$$

and

$$\mid n,-\rangle = \begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\varphi}/2 \\ \cos\frac{\theta}{2}e^{-\varphi/2}\end{pmatrix} \qquad (4)$$

Using these one can show that $P(\hat{n},\pm) = \frac{1}{2}(1\pm\hat{n}\cdot\vec{\sigma})$ is the projection operator for $\mid n,\pm\rangle$ states. In particular

$$\frac{1}{2}(1+\hat{n}\cdot\vec{\sigma}) \mid \sigma_z=+1\rangle = \cos\frac{\theta}{2}e^{i\varphi/2} \mid n,+\rangle \qquad (5a)$$

$$\frac{1}{2}(1+\hat{n}\cdot\vec{\sigma}) \mid \sigma_z=-1\rangle = \sin\frac{\theta}{2}e^{-i\varphi/2} \mid n,+\rangle \qquad (5b)$$

$$\frac{1}{2}(1-\hat{n}\cdot\vec{\sigma}) \mid \sigma_z=-1\rangle = \cos\frac{\theta}{2}e^{-i\varphi/2} \mid n,-\rangle \qquad (5c)$$

$$\frac{1}{2}(1-\hat{n}\cdot\vec{\sigma}) \mid \sigma_z=+1\rangle = -\sin\frac{\theta}{2}e^{i\varphi/2} \mid n,-\rangle \qquad (5d)$$

Let us consider the expectation value of projection on to $\mid \hat{n},\pm\rangle$and $\mid \hat{m},\pm\rangle$ when acted on the state $\mid\psi^-\rangle$ above. Define $P^A(\hat{n},\pm) = \frac{1}{2}(1\pm\hat{n}\cdot\sigma)$ and $P^B(\hat{m},\pm) = \frac{1}{2}(1\pm\hat{m}\cdot\sigma)$

, where the directions $\hat{n}$ and $\hat{m}$ makes angles θ_1 and θ_2 respectively, with the z axis. We have,

$$\begin{aligned} P^A(\hat{n},+)P^B(\hat{m},+) \mid \psi^-\rangle &= P^A(\hat{n},+)P^B(\hat{m},+)\left[\frac{1}{\sqrt{2}}\left(\mid \sigma_z^A=+,\sigma_z^B=-\rangle - \mid \sigma_z^A=-,\sigma_z^B=+\rangle\right)\right] \\ &= \frac{1}{\sqrt{2}}\left[\cos\frac{\theta_1}{2}e^{i\varphi/2}\sin\frac{\theta_2}{2}e^{-i\varphi/2}\mid n+,m+\rangle - \sin\frac{\theta_1}{2}e^{-i\varphi/2}\cos\frac{\theta_2}{2}e^{i\varphi/2}\mid n+,m+\rangle\right] \\ &= \frac{1}{\sqrt{2}}\sin\left(\frac{\theta_2-\theta_1}{2}\right)\mid n+,m+\rangle = \frac{1}{\sqrt{2}}\sin\left(\frac{\theta}{2}\right)\mid n+,m+\rangle \end{aligned}$$

where $\theta = \theta_2 - \theta_1$ is the angle between the axes $\hat{n}$ and $\hat{m}$.
In a similar way, one can show that

$$P^A(\hat{n},+)P^B(\hat{m},-) = \frac{1}{\sqrt{2}}\cos\left(\frac{\theta}{2}\right)\mid n+,m-\rangle$$

Since $P^2 = P$, one can get identical expressions by acting identical operator to the bras to the left and get

$$\langle\psi^- \mid P^A(\hat{n},+)P^B(\hat{m},+)\mid\psi^-\rangle = \langle\psi^- \mid P^A(\hat{n},-)P^B(\hat{m},-)\mid\psi^-\rangle = \frac{1}{2}\sin^2\frac{\theta}{2} = \frac{1}{4}(1-\cos\theta) \tag{6a}$$

$$\langle\psi^- \mid P^A(\hat{n},+)P^B(\hat{m},-)\mid\psi^-\rangle = \langle\psi^- \mid P^A(\hat{n},-)P^B(\hat{m},+)\mid\psi^-\rangle = \frac{1}{2}\cos^2\frac{\theta}{2} = \frac{1}{4}(1+\cos\theta) \tag{6b}$$

It can be checked that since $\langle\hat{n}\cdot\sigma\rangle = 0$, (**??**) gives (**??**). Till now our derivation has not specified the directions $\hat{n}$ and $\hat{m}$. Let us close Alice's direction $\hat{n}$ to be one of the three directions

$$\begin{aligned} \hat{n}_1 &= \hat{z} \\ \hat{n}_2 &= \frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{z} \\ \hat{n}_3 &= -\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{z} \end{aligned}$$

Of course, once Alice makes her measurement along any of these directions, she would have disturbed the state of the spin and will not be in a position to find out what would have been the result if she had chosen a different axis. Suppose, however, Bob measures along $\hat{n}_1$ and communicates his result to Alice, then Alice would know what would happen if she measured along $\hat{n}_1$ as their measurements must be anti- correlated. So, instead, Alice measures along $\hat{n}_2$ and gets a result. The probability that her measurements along $\hat{n}_1$ and $\hat{n}_2$ give the same result (i.e. if she gets + along n_1, she also gets + along n_2 and likewise if she gets − along n_1 she also gets − along n_2) is equal to the correlation between

Alice and Bob getting same results when Alice measures along $\hat{n}_2$ and Bob along $-\hat{n}_1$. and get the same results. The probability is (two results, ++ and −−)

$$P_{same} = 2 \times \frac{1}{4}(1 - \cos\theta) = \frac{1}{2}(1 - \cos\theta) \tag{7}$$

In this case θ is the angle between $\hat{n}_1$ and $-\hat{n}_2$, which is $\cos\theta = 1/2$ (This is because Bob makes the measurement along $-n_2$ in order that his measurement corresponds to the result Alice would have got had she measured along n_2). We can choose any two directions and obtain such results. Thus if Alice's measurements are obtained for $\hat{n}_2$ and $\hat{n}_3$ (i.e., Bob $-\hat{n}_3$), we have $\cos\theta = 1/2$ and for $\hat{n}_2$ and $\hat{n}_3$, $\cos\theta = 1/2$. In each case, the probability that the the pair of measurement gave the *same* result is $\frac{1}{4}$, which is obtained by putting $\cos\theta = \frac{1}{2}$ in (**??**) above.Thus as per quantum mechanics, the probability of getting the same result for three pairs of axes chosen as above is

$$P(\hat{n}_1, \hat{n}_2) + P(\hat{n}_2, \hat{n}_3) + P(\hat{n}_3, \hat{n}_1) = 3 \times \frac{1}{4} = \frac{3}{4} < 1$$

If, however, there were hidden variables in the problem, the probabilities arise from a statistical distribution of the probabilities of different possible results. (Since Alice does not measure her bit in the same direction as Bob does, the collapsing of the state as a result of Bob's measurement does not specify what result Alice would get as a result of her measurement.)Remember that as there are only three experiments, at least in one of the pairs the result has to be the same, so that the probability of at least one pair of experiments is 1, thereby giving

$$P(\hat{n}_1, \hat{n}_2) + P(\hat{n}_2, \hat{n}_3) + P(\hat{n}_3, \hat{n}_1) > 1 \tag{8}$$

(Since the measurement along each axis gives either a + or − and the result of the measurement along any axis is not influenced by measurement along another axis, the result is independent. Since there are only two possibilities, measurement along at least one pair has to be the same)

Equation (**??**) is one of the inequalities which can provide a test for quantum mechanics.

4 CHSH Inequality

Bell's inequalities essentially provide a test for the validity of quantum mechanics agains the counter claims of EPR regarding the existence of hidden variables. If quantum mechanics is correct, the Bell's inequalities are violated. One of these is known as CHSH (Clauser, Horne, Shimony and Holt) inequality.
Suppose Charlie prepares two particles in a singlet state ψ^- as in the previous section.

Alice controls the first qubit and Bob the second. Each of them can measure two properties of their particle, Alice measuring properties P_Q and P_R and Bob measuring properties P_S and P_T. Result of each of these experiments can be either +1 or −1.
A common place classical similarity is that Alice and Bob have been given one each of a pair of T-shirts. Alice can observe its color, which can be blue (+1) or red (−1) and its size, which can be large (+1) or medium (−1). Bob can observe its price, which can be high (+1) or average (−1) and the quality of its fabric, high (+1) or average (−1).
The result of measurement are denoted by Q, R, S, T. These are object properties which can be revealed by actual observation. In order to decide, what property he will measure Bob tosses a coin and measure the property which is decided by whether coin gives head or tail, i.e. the choice of property is random. Alice measures the properties of her particle in a random fashion as well but does it precisely at the same time as Bob does his measurement so that the measurements are not causally related. Since each measurement returns ±1, after a series of measurements on a large number of replicas, a table of measurement of each pair is prepared. Thus all those measurements where Bob measured P_S and Alice measured P_Q are tabulated. Similar tables are made for other possible pairs and groups of 4 pairs of measurements are made. Consider the expression

$$QS + RS + RT - QT = (Q + R)S + (R - Q)T$$

Note that in the above expression, one of the terms must be zero because either $Q+R = 0$ or $R - Q = 0$. When one of these vanishes, the other one has a value ±2. Thus we have,

$$QS + RS + RT - QT = \pm 2$$

Let us look at what would happen if there was an a-priory probability of the measurement giving +1 or −1. This would happen if there were hidden variables in the system.Let $p(q, r, s, t)$ be the probability that *before measurement* the system is in the state $Q = q, R = r, S = s$ and $T = t$. We then have

$$\begin{aligned} \langle QS + RS + RT - QT \rangle &= \sum_{q,r,s,t} (qs + rs + rt - qt) \\ &\leq \sum_{p,q,r,s} \times 2 = 2 \end{aligned} \tag{9}$$

However,

$$\langle QS + RS + RT - QT \rangle = \langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle$$

which gives **CHSH inequality**

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle \leq 2 \tag{10}$$

The above inequality has been obtained under the assumption of existence of hidden variables, i.e. under assumptions of local realism.

We will now calvculate the left hand side of (**??**) under quantum mechanics. We have two particles which are in a singlet state and Alice and Bob would make projective measurements. Suppose the measurements are taken to be single qubit gates defined by

$$Q = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

$$R = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

$$S = \frac{-Z - X}{\sqrt{2}} Z = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \tag{13}$$

$$T = \frac{Z - X}{\sqrt{2}} Z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \tag{14}$$

It is straightforward to get the following relations

$$\begin{aligned}
Q \mid 0\rangle &=\mid 0\rangle \\
Q \mid 1\rangle &= -\mid 1\rangle \\
R \mid 0\rangle &=\mid 1\rangle \\
R \mid 1\rangle &= -\mid 0\rangle \\
S \mid 0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -\frac{\mid 0\rangle + \mid 1\rangle}{\sqrt{2}} \\
S \mid 1\rangle &= \frac{-\mid 0\rangle + \mid 1\rangle}{\sqrt{2}} \\
T \mid 0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}} \\
T \mid 1\rangle &= -\frac{\mid 0\rangle + \mid 1\rangle}{\sqrt{2}}
\end{aligned}$$

Using these, we can immediately calculate each term on the left of (**??**). We have

$$\begin{aligned}
\langle QS\rangle &= \frac{1}{2}\left[\langle 01 \mid QS \mid 01\rangle - \langle 01 \mid QS \mid 10\rangle - \langle 10 \mid QS \mid 01\rangle + \langle 10 \mid QS \mid 10\rangle\right] \\
&= 9\frac{1}{2}\left[1 \times \frac{1}{\sqrt{2}} - 0 - 0 + (-1) \times \frac{-1}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}} \\
\langle RS\rangle &= \frac{1}{\sqrt{2}} \\
\langle RT\rangle &= \frac{1}{\sqrt{2}} \\
\langle QT\rangle &= -\frac{1}{\sqrt{2}}
\end{aligned}$$

Substituting the above in (**??**), we get

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2} \tag{15}$$

which violates the Bell's inequality (**??**). Experimental tests have been devised to verify Bell's inequalities and it has been found to be always violated, thereby providing support for quantum mechanics over local hidden variable theory.