Topic -6
No-Cloning Theorem and Quantum Teleportation

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1 Introduction

In the last lecture, we talked about various quantum circuits. We had discussed the essential elements of a quantum circuit which consists of input and out registers, some ancilla bits, gates representing unitary operators, oracles and measuring apparatus. We pointed out that the measurement has a special place in quantum computing, as unlike classical computers, measurement in a quantum computer gives one of many possible outputs with a certain probability. One of the tasks of a quantum computer is to extract relevant information from such probabilistic measurements. We also discussed realization of several classical logics using quantum processes.
In this lecture we will discuss two aspects of quantum computing and circuits which are essentially quantum phenomenon with no classical analogue. These are quantum No-cloning theorem and teleportation.

2 Quantum No-Cloning Theorem

Before discussing the theorem, consider what is meant by copying something, for instance in a device such as a Xerox machine. We need the original to be copied, a blank paper on which to make a copy of the original, run the pair through the machine to get back the original and an identical copy of the same in place of the blank used. We use the same concept in defining copy of a quantum state. We would start with the state to be copied, a standard state (a blank) on which the machine would copy the original state to get back the original state and a copy in place of the blank state. Since quantum operations are unitary, we ask the question, does a unitary operator exist which acting on the former pair gives the desired output? Stated mathematically, can one find an operator U such that we have

$$U \mid \psi\rangle \otimes \mid s\rangle = \mid \psi\rangle \otimes \mid \psi\rangle ?$$

In the above $\mid s\rangle$ stands for the standard or the blank state, which could, for instance, be the null state $\mid 0\rangle$. One must make an observation here. Like in our analogy of the Xerox machine, if such an operator does exist, it should be able to clone different states and not just be special for some particular state just as we do not have different Xerox machines for copying different originals.
Supposing such cloning were possible and there existed such an operator U. We would then have corresponding to its application on two different states $\mid \psi\rangle$ and $\mid \phi\rangle$,

$$U \mid s\rangle \otimes \mid \psi\rangle = \mid \psi\rangle \otimes \mid \psi\rangle$$
$$U \mid s\rangle \otimes \mid \phi\rangle = \mid \phi\rangle \otimes \mid \phi\rangle$$

If this were possible, we would have

$$\begin{aligned}\langle \psi \mid \phi\rangle &= \langle \psi \mid (\langle s \mid s\rangle \mid \phi\rangle \\ &= \langle \psi, s \mid s, \phi\rangle \\ &= \langle \psi, s \mid U^\dagger U \mid s, \phi\rangle\end{aligned}$$

wherein in the above, in the first line we have introduced an identity $\langle s \mid s\rangle$ using the fact that the standard states are normalized, in the second line we have rewritten the above using the two qubit notation and in the last line, we have formally introduced the operator U stated above which by its unitarity satisfies $U^\dagger U = 1$. We now let the operator U act to its right and $U^\dagger$ act to its left, giving

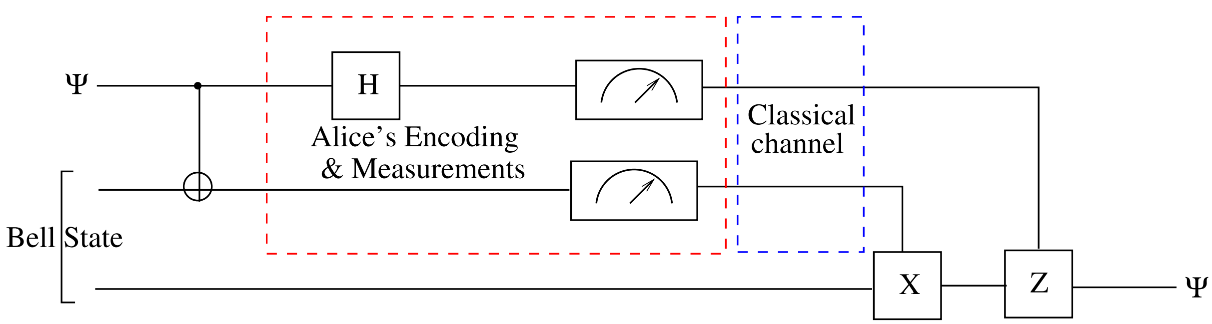
$$\langle \psi \mid \phi\rangle = \langle \psi, \psi \mid \phi, \phi\rangle = \langle \psi \mid \phi\rangle \times \langle \psi \mid \phi\rangle = \langle \psi \mid \phi\rangle^2$$

where the ordering of the scalar product is immaterial. This shows that the scalar product of the states $\mid \psi\rangle$ and $\mid \phi\rangle$ must be either 1 or zero- in other words, they should be either the same state or be two orthogonal states. This shows, there does not exist a unitary operator which can clone arbitrary, non-orthogonal states. This theorem has significance in various aspects of quantum communication. For instance, we will see in later lectures that one of the standard protocols in classical communication is to transmit multiple copies of a message so that the error during communication is minimized. No-Cloning theorem makes this protocol not useful in quantum communication. In the following we discuss a communication protocol in which an arbitrary state is to be sent to a receiver without a physical transfer of the state from sender to the receiver.

3 Quantum Teleportation

Teleportation has been for long in the folklore of science fiction. In the television serial Star-Trek, one finds the captain of the ship Captain Kirk being energized at one place and reassembled in material form in another place. The object of teleportation is to transfer a state from one point to another without its having travelled continuously between the two points. This would not be possible in classical communication. In quantum communication we use a protocol between the sender Alice and receiver Bob, who share an entangled pair, which we will take as one of the four Bell states. Alice, in addition, has in her possession an arbitrary single qubit quantum state $\alpha \mid 0\rangle + \beta \mid 1\rangle$ which she intends to be transported to Bob. Alice can only perform local operations on her qubits and Bob on his. They may use classical channel for communicating result of a measurement but not physically transport a state. However, Alice cannot give such information to Bob describing the state she wants Bob to have a copy of, as such description would, in general, require infinite amount of information transfer as the coefficients α and β are complex and exact description of these two complex numbers with possibly irrational real and imaginary parts, require infinite amount of binary information. Further, in case Alice does not know the exact values of α and β, she herself cannot get these by measuring locally as this would make the state collapse to one of the basis states obliterating such information.

Let us assume that Alice and Bob share one qubit each of the Bell state $\mid \Psi\rangle = \frac{1}{\sqrt{2}}(\mid 00\rangle + \mid 11\rangle)$ which is an entangled state. We will assume that Alice has the first qubit of this pair and Bob the second. Since there are in all three qubits, two with Alice and one with Bob, we will number these as follows: the qubit which needs to be transported $\alpha \mid 0\rangle + \beta \mid 1\rangle$ will be labelled as qubit 1, Alice's part of the entangled state as qubit 2 and Bob?s qubit as bit 3. Remember that the qubits 2 and 3 are entangled. The teleportation circuit is shown in the diagram below.



Alice makes the state $\mid \Psi\rangle$ interact with her part of the entangled qubit by first doing a CNOT operation with the state $\mid \Psi\rangle$ (the first qubit) as the control and the qubit 2 as the target. This will make the three qubits transform as follows.

$$CNOT\left[(\alpha \mid 0\rangle + \beta \mid 1\rangle) \otimes \frac{1}{\sqrt{2}}(\mid 0\rangle \otimes \mid 0\rangle + \mid 1\rangle \otimes \mid 1\rangle)\right] = \frac{1}{\sqrt{2}}\left[\alpha\,(\mid 000\rangle + \mid 011\rangle) + \beta\,(\mid 110\rangle + \mid 101\rangle)\right]$$

where in the above, the qubits are numbered as indicated above. Alice now subjects the first qubit to a Hadamard gate. Remembering that Hadamard gate would transform $\mid 0\rangle$

to $\frac{1}{\sqrt{2}}(\mid 0\rangle + \mid 1\rangle)$ and $\mid 1\rangle$ to $\frac{1}{\sqrt{2}}(\mid 0\rangle - \mid 1\rangle)$. We would then get

$$\frac{1}{2}\left[\alpha\left(\mid 000\rangle + \mid 100\rangle + \mid 011\rangle + \mid 111\rangle\right) + \beta\left(\mid 010\rangle - \mid 110\rangle + \mid 001\rangle - \mid 101\rangle\right)\right]$$

We rearrange the terms by writing Alice?s qubits separately from Bob?s qubit. Other than for an overall factor of 1/2, the above state is

$$\mid 00\rangle(\alpha \mid 0\rangle + \beta \mid 1\rangle) + \mid 01\rangle(\alpha \mid 1\rangle + \beta \mid 0\rangle) + \mid 10\rangle(\alpha \mid 0\rangle - \beta \mid 1\rangle) + \mid 11\rangle(\alpha \mid 1\rangle - \beta \mid 0\rangle)$$

If now, Alice makes a measurement of her qubits (both first and the second qubits) she will get four possible results $\mid 00\rangle, \mid 01\rangle, \mid 10\rangle$ and $\mid 11\rangle$ with equal probability of 1/4 each. The result of her measurement can be encoded as two classical bits 00, 01,10 and 11 respectively. She can now communicate to Bob, using a classical channel, the result of her measurement using such classical bits. On getting information from her, Bob would initiate some processing at his end to convert his qubit to the qubit Alice wanted him to have. Remembering that Bob's qubit would have collapsed to the state multiplying Alice?s qubits corresponding to her measurement, these actions are as follows:

1. If Alice's measurement gives 00, Bob already has the state $\mid \Psi\rangle$. In this case he does nothing.

2. If Alice's measurement gives 01, Bob's state has collapsed to the state $\alpha \mid 1\rangle + \beta \mid 0\rangle$. In such a case Bob applies an X-gate to his qubit which will interchange the states $\mid 0\rangle$ and $\mid 1\rangle$ and convert Bob's state to the desired form.

3. If Alice's result is 10, Bob's state has collapsed to the state $\alpha \mid 0\rangle - \beta \mid 1\rangle$. In such a case Bob applies a Z-gate on his qubit to get $\mid \Psi\rangle$. Finally,

4. If Alice's result is 11, Bob's state has collapsed to the state $\alpha \mid 1\rangle - \beta \mid 0\rangle$. In such a case Bob first applies a X-gate converting his state to $\alpha \mid 0\rangle - \beta \mid 1\rangle$. He now applies an Z-gate to recover $\mid \Psi\rangle$, the desired state.
   There are two questions that arise in the above. Firstly, nowhere in our discussion we mentioned what is the distance between Alice and Bob; they could, for instance, be separated by space-like distances, in which case we could be violating the principle of relativity whereby communication with a speed greater than that of light is not possible. However, recall that in the teleportation protocol, there is a step in which Alice must communicate result of her measurement to Bob using a classical channel. This is obviously not possible with a speed greater than that of light. Second question that arises is have we, in the process, copied a quantum state, in violation of the No-Cloning theorem? The answer again is no because if one looks at what Alice has with her in place of the state $\mid \Psi\rangle$, one finds that she has either the state $\mid 0\rangle$ or the state $\mid 1\rangle$. Likewise Bob has also lost his original qubit while reconstructing the state $\mid \Psi\rangle$.