Quantum Information and Computing-Topic- 7 :Super Dense Coding

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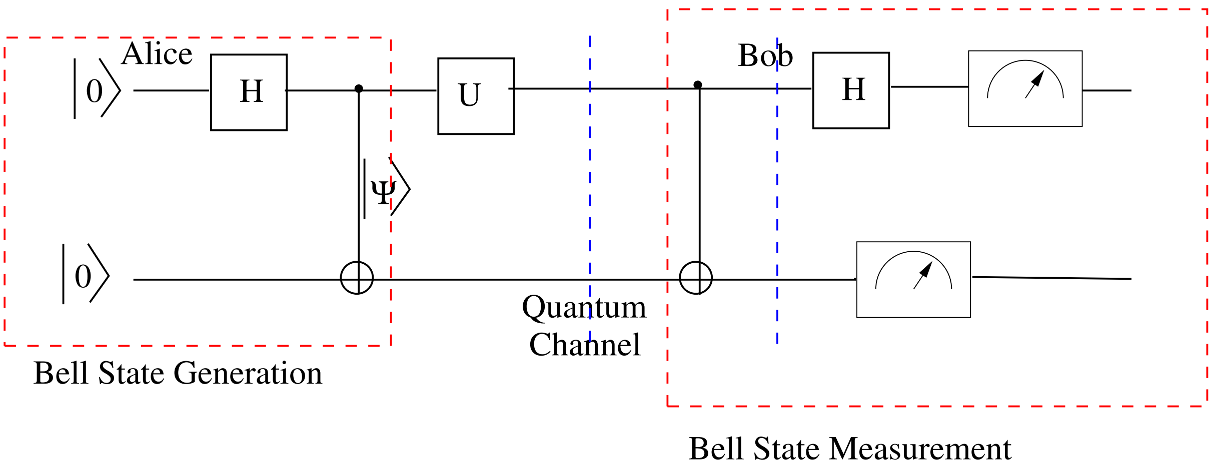
1 Introduction

In the last lecture we introduced the principle of teleportation of a quantum state from one place to another by means of a protocol which required Alice to send two bits of classical information to Bob. This enabled Bob to do some local operation on his part of a shared qubit to reconstruct a quantum state which Alice had to begin with.
In this lecture we will talk about **Super Dense Coding** whereby Alice, who has with her two bits of classical information will be able to provide Bob about her bits by sending one bit of quantum information. It may be realized that there is no possible way in which two classical bits may be encoded into a single bit. The task will be made possible by an entangled pair which Alice and Bob shared. Alice, as in the case of quantum teleportation, can perform local operations on her qubit of the entangled pair. As her qubit is entangled with that of Bob, any measurement she performs will have an effect on Bob's qubit as well. We will see in this lecture, that given this previous engagement between Alice and Bob, it will be possible for Alice to let Bob have information about her two cbits by trading only one bit of quantum information.

2 Dense Coding Circuit

As in the case of quantum teleportation, we assume that Alice and Bob have one qubit each of an entangled state, which we take to be one of the four Bell states. In this protocol, Alice will perform certain local operations on her qubit, depending on the two cbits she is interested in Bob to have information about. Having done this she will send her qubit to Bob through a quantum channel. Thus she will be only sending a single bit of quantum state instead of two classical bits. Bob, when he gets hold of Alice's part of the entangled qubit (which has now been modified by Alice) as well will perform certain operations at his end to get the desired cbits. The quantum circuit of super dense coding is shown in the figure.



In the figure, the extreme left section is not a part of dense coding circuit as it simply is used to generate the Bell state which Alice and Bob share. In this example, the Bell state that we have chosen $\frac{1}{2}(|\,00\rangle + |\,11\rangle)$. The following table gives the action (the unitary operation represented by U in the figure) that Alice would take on her qubit, depending on the cbits she wants to send to Bob.

cbit to be sent	Action by Alice	Resulting state
00	No action by Alice	$\frac{1}{\sqrt{2}}(\mid 00\rangle + \mid 11\rangle)$
01	Alice applies $X \otimes I$	$\frac{1}{\sqrt{2}}(\mid 10\rangle + \mid 01\rangle)$
10	Alice applies $iY \otimes I$	$\frac{1}{\sqrt{2}}(- \mid 10\rangle + \mid 01\rangle)$
11	Alice applies $Z \otimes I$	$\frac{1}{\sqrt{2}}(\mid 00\rangle - \mid 11\rangle)$

Note that what has happened as a result of Alice's operation is that corresponding to each of the four cbits she wants to send Bob, it has converted the original Bell state to four different states of the Bell basis.
Now that Bob has possession of the entangled pair, he will have to determine what state he has. In order to determine the state she has, Bob applies a CNOT gate using the qubit which was originally with Alice, i.e. the first qubit. The result of CNOT operation be as follows:

cbit to be sent	state Bob has	Result of CNOT
00	$\frac{1}{\sqrt{2}}(\mid 00\rangle + \mid 11\rangle)$	$\frac{1}{\sqrt{2}}(\mid 00\rangle + \mid 10\rangle)$
01	$\frac{1}{\sqrt{2}}(\mid 10\rangle + \mid 01\rangle)$	$\frac{1}{\sqrt{2}}(\mid 11\rangle + \mid 01\rangle)$
10	$\frac{1}{\sqrt{2}}(- \mid 10\rangle + \mid 01\rangle)$	$\frac{1}{\sqrt{2}}(- \mid 11\rangle + \mid 01\rangle)$
11	$\frac{1}{\sqrt{2}}(\mid 00\rangle - \mid 11\rangle)$	$\frac{1}{\sqrt{2}}(\mid 00\rangle - \mid 10\rangle)$

It may be noted that application of CNOT has resulted in expressing the two qubits as product of two single qubits, thereby removing the entanglement that existed before the application of CNOT gate. In the product form, the state which Bob has corresponding to the four cases are as follows:

cbit	State
00	$\frac{\mid 0\rangle + \mid 1\rangle}{\sqrt{2}} \otimes \mid 0\rangle$
01	$\frac{\mid 0\rangle + \mid 1\rangle}{\sqrt{2}} \otimes \mid 1\rangle$
10	$\frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}} \otimes \mid 1\rangle$
11	$\frac{\mid 0\rangle - \mid 1\rangle}{\sqrt{2}} \otimes \mid 0\rangle$

If Bob now measures the second qubit, in the first and the fourth case, he will get the state $\mid 0\rangle$ while in the second and the third, he will get the state $\mid 1\rangle$. However, it may be noted that the two cases which give the same result for the second qubit, the first qubit happens to be in different linear combination of the states $\mid 0\rangle$ and $\mid 1\rangle$. The first qubit, if sent through a Hadamard gate before measurement, will decide the cbits which Alice had intended to be sent.

cbit	Final State
00	$\mid 0\rangle \otimes \mid 0\rangle$
01	$\mid 0\rangle \otimes \mid 1\rangle$
10	$\mid 1\rangle \otimes \mid 1\rangle$
11	$\mid 1\rangle \otimes \mid 0\rangle$

It may be remarked that though we have used Bell states to illustrate how super dense coding works, the protocol is not really restricted to the Bell set. Suppose we have a basis set S of n qubits whose members are $s_0, s_1, \ldots s_{2^n-1}$. Remember that in our illustration, Alice subjected her part of the entangled pair to different unitary gate depending on the cbits she wanted Bob to have. In the general case, if there exists a unitary operation corresponding to each member of the set which is such that acting on a member of the set, it would convert that member into another member of the same set, i.e. if there exists a set $U = \{U_0^i, U_1^i, \ldots, U_{2^n-1}^i\}$ such that $U_j^i s_i = s_j$, then the dense coding would work.
As an illustration consider a three qubit entangled state known as a GHZ state, which is a member of an 8 element set. Of the chosen entangled state, Alice has in her possession two qubits and Bob the third. Alice wants to encode one of the 8 cbits having three digits, viz., 000, 001, 010, 011, 100, 101, 110 and 111. In the super dense coding protocol, she will perform some local operations on her two qubits of this entangled state, converting the GHZ state to a unique member of the set corresponding to different values of the cbits. For instance, if Alice and Bob originally had $\frac{1}{\sqrt{2}}(\mid 000\rangle + \mid 111\rangle)$, Alice applies the following unitary operation on her qubits (the identity operator for Bob's qubit is omitted in the following)

cbit to be sent	Action by Alice	Resulting state
000	No action by Alice	$\frac{1}{\sqrt{2}}(\mid 000\rangle + \mid 111\rangle)$
001	Alice applies $Z \otimes I$	$\frac{1}{\sqrt{2}}(\mid 000\rangle - \mid 111\rangle)$
010	Alice applies $X \otimes I$	$\frac{1}{\sqrt{2}}(\mid 100\rangle + \mid 011\rangle)$
011	Alice applies $iY \otimes I$	$\frac{1}{\sqrt{2}}(- \mid 100\rangle + \mid 011\rangle)$

cbit to be sent	Action by Alice	Resulting state
100	Alice Applies $I \otimes X$	$\frac{1}{\sqrt{2}}(\mid 010\rangle + \mid 101\rangle)$
101	Alice applies $Z \otimes X$	$\frac{1}{\sqrt{2}}(\mid 010\rangle - \mid 101\rangle)$
110	Alice applies $X \otimes X$	$\frac{1}{\sqrt{2}}(\mid 110\rangle + \mid 001\rangle)$
111	Alice applies $iY \otimes X$	$\frac{1}{\sqrt{2}}(- \mid 110\rangle - \mid 001\rangle)$

The analysis of Bob's measurement has a pattern similar to that discussed for the 2 cbit transfer and is left as an exercise.